# Solving Complex Quadratic Equations with Full-rank Random Gaussian Matrices

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## System of quadratic equations

- Quadratic measurements obtained with high-rank measurement matrices arise in applications such as unassigned distance geometry problem.
- Most prior works focus on rank-1 psd measurement matrices or real measurements.
- **Measurement Model**:

$$y_i = \boldsymbol{x}^* \boldsymbol{A}_i \boldsymbol{x}, \quad i = 1, \cdots, m\,. \tag{1}$$

  - $\boldsymbol{x} \in \mathbb{C}^n$ is the complex signal.
  - $y_i \in \mathbb{C}$ is the $i$-th complex quadratic measurement.
  - $\boldsymbol{A}_i \in \mathbb{C}^{n\times n}$ is the $i$-th complex random Gaussian measurement matrix.

## Problem formulation

We minimize the following objective function $f(\boldsymbol{z})$:

$$f(\boldsymbol{z}) = \frac{1}{m}\sum_{i=1}^{m} \left|\boldsymbol{z}^* \boldsymbol{A}_i \boldsymbol{z} - y_i\right|^2, \tag{2}$$

using gradient descent:

$$\boldsymbol{z}^{(t+1)} = \boldsymbol{z}^{(t)} - \eta \nabla f(\boldsymbol{z}), \tag{3}$$

where $\eta > 0$ is the step size.

- Nonconvex optimization problem.
- $\boldsymbol{x}e^{\boldsymbol{j}\phi}$ is a global minimum solution for all $\phi \in [0, 2\pi)$.
- The distance between the recovered $\boldsymbol{z}$ and a global minimum solution $\boldsymbol{x}$ is

$$\operatorname{dist}(\boldsymbol{z}, \boldsymbol{x}) = \min_{\phi\in[0,2\pi)} \left\|\boldsymbol{z} - \boldsymbol{x}e^{\boldsymbol{j}\phi}\right\|_2 \tag{4}$$

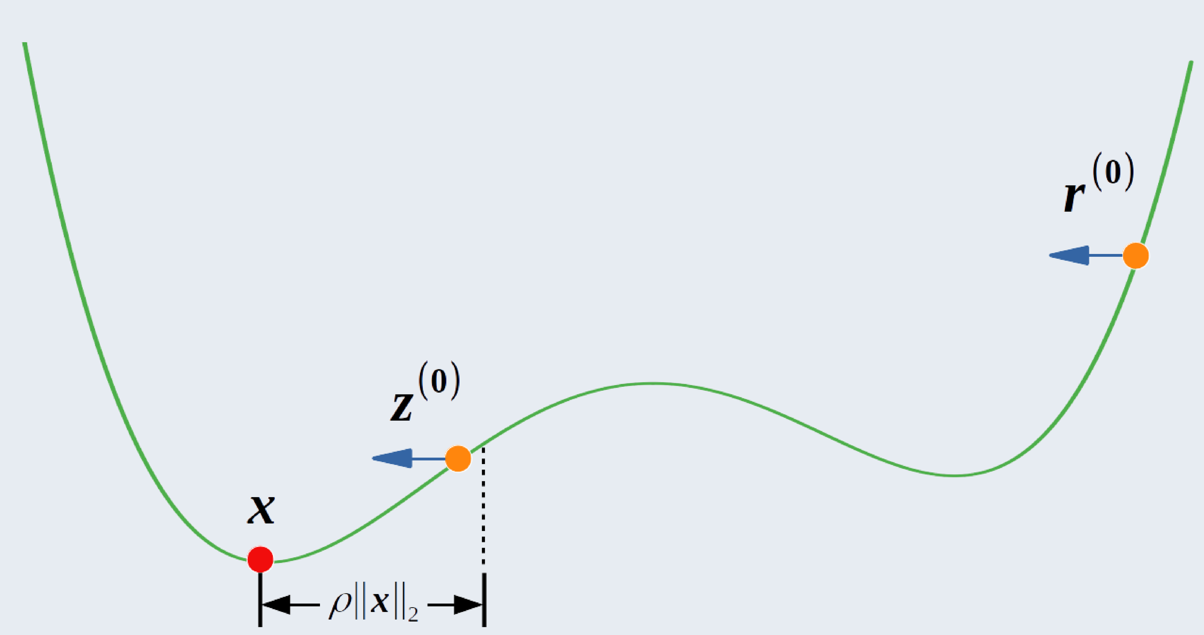



Figure 1: A good initialization is needed to solve a nonconvex optimization problem via gradient descent.

## Spectral initialization

The left or right singular vector $\boldsymbol{z}^{(0)}$ of the following matrix:

$$\boldsymbol{S} = \frac{1}{m}\sum_{i=1}^{m} \overline{y}_i \cdot \boldsymbol{A}_i = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{x}^* \boldsymbol{A}_i^* \boldsymbol{x} \cdot \boldsymbol{A}_i\,. \tag{5}$$

- For sufficiently large $m$, with high probability $\boldsymbol{S}$ concentrates around its expectation in terms of spectral norm.

$$\mathbb{E}[\boldsymbol{S}] = 2\boldsymbol{x}\boldsymbol{x}^* \tag{6}$$

- The spectral initializer $\boldsymbol{z}^{(0)}$ is close to a global optimum $\boldsymbol{x}$ [1,2].

$$\operatorname{dist}(\boldsymbol{z}^{(0)}, \boldsymbol{x}) \le \delta \|\boldsymbol{x}\|_2 \tag{7}$$

holds with high probability.

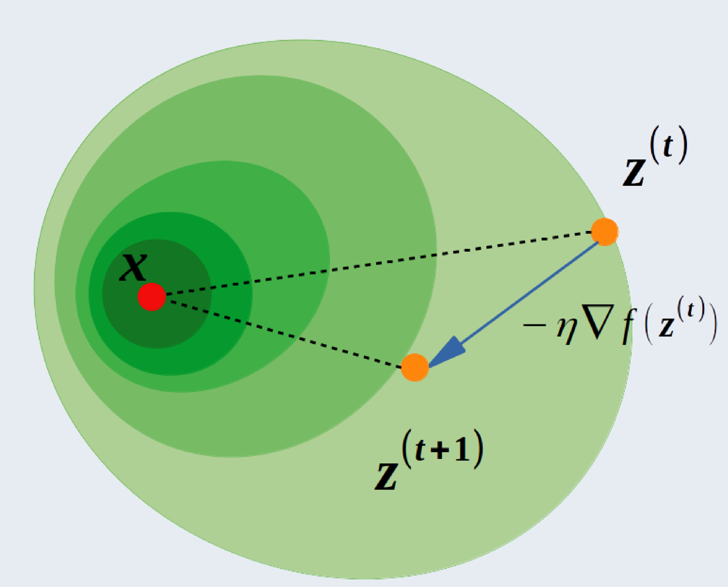
### Lemma

*When $m \ge Cn$ for some sufficiently large constant $C$, for all $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{C}^n$ satisfying $\|\boldsymbol{p}\|_2 = 1$, $\|\boldsymbol{q}\|_2 = 1$ and every $\nu > 0$, the following*

$$\left\|\frac{1}{m}\sum_{i=1}^{m} \boldsymbol{p}^* \boldsymbol{A}_i^* \boldsymbol{q} \cdot \boldsymbol{A}_i - 2\boldsymbol{q}\boldsymbol{p}^*\right\| \le \nu\,, \tag{8}$$

*holds with high probability.*

## Linear convergence




$$\operatorname{dist}(\boldsymbol{z}^{(t+1)}, \boldsymbol{x}) \le \mu \cdot \operatorname{dist}(\boldsymbol{z}^{(t)}, \boldsymbol{x}), \quad \mu \in (0, 1) \tag{9}$$

- Guaranteed if $f(\boldsymbol{x})$ satisfies the regularity condition $RC(\alpha, \beta, \rho)$ [2] for all $\boldsymbol{z} \in E(\rho) = \{\boldsymbol{z} \mid \operatorname{dist}(\boldsymbol{z}, \boldsymbol{x}) \le \rho\|\boldsymbol{x}\|_2\}$.

$$\operatorname{Re}\left(\langle \nabla f(\boldsymbol{z}), \boldsymbol{z} - \boldsymbol{x}e^{\boldsymbol{j}\phi_{\min}}\rangle\right) \ge \frac{1}{\alpha}\operatorname{dist}^2(\boldsymbol{z}, \boldsymbol{x}) + \frac{1}{\beta}\|\nabla f(\boldsymbol{z})\|_2^2, \tag{10}$$

where $\alpha > 0,\ \beta > 0,\ 1 > \rho > 0$ are some constants.

## Main theorem

*When $m \ge Cn$ for some sufficiently large constant $C$,*

1. *There exists a choice of $1 > \nu > 0$, $1 > \rho > 0$, $\alpha > 0$, $\beta > 0$ such that $RC(\alpha, \beta, \rho)$ holds on $E(\rho)$ with high probability.*
2. *Furthermore, if the step size $0 < \eta \le \frac{2}{\beta}$, the gradient descent with the spectral initializer $\boldsymbol{z}^{(0)}$ converges linearly to $\boldsymbol{x}$*

$$\operatorname{dist}\left(\boldsymbol{z}^{(t)}, \boldsymbol{x}\right) \le \left(\sqrt{1 - \frac{2\eta}{\alpha}}\right)^{t} \rho \cdot \|\boldsymbol{x}\|_2\,, \tag{11}$$

*with high probability.*

## Experimental results

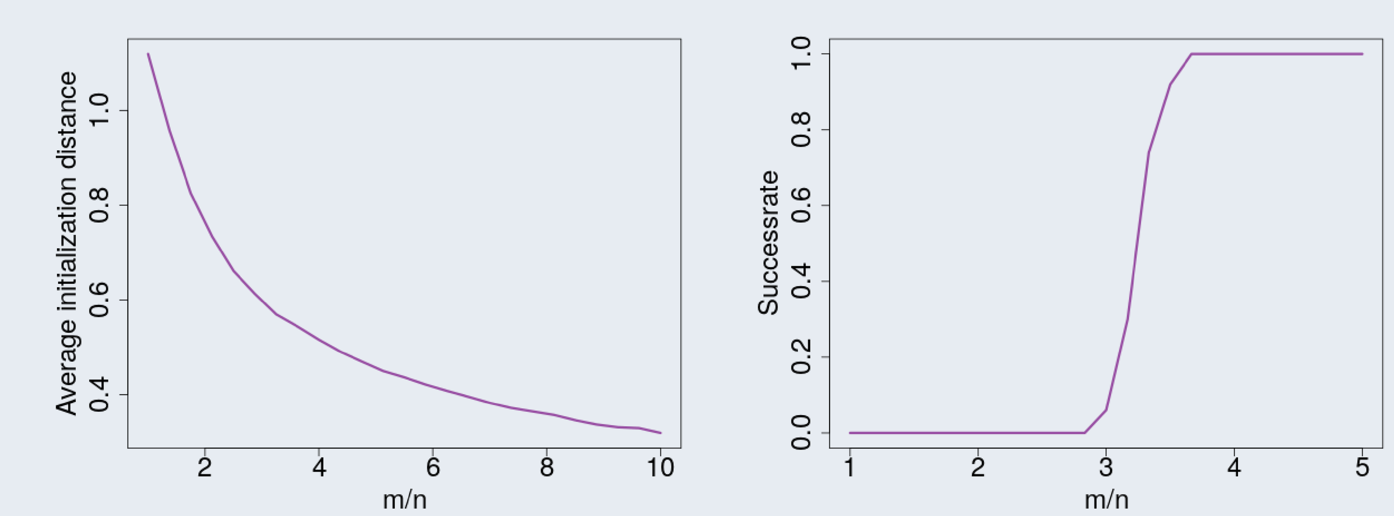



Figure 2: Left: $\operatorname{dist}\left(\boldsymbol{z}^{(0)}, \boldsymbol{x}\right)$; Right: Phase transition of the success rate.

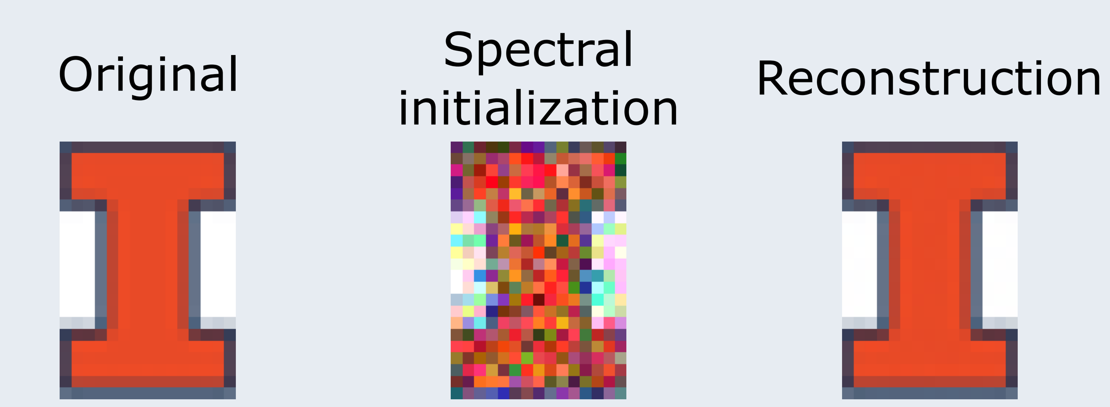



Figure 3: Recovery of the UIUC logo.

## References

[1] Netrapalli et al., Phase retrieval using alternating minimization, IEEE Trans. Signal Process., 2015

[2] Candès et al., Phase retrieval via Wirtinger flow: Theory and algorithms, IEEE Trans. Inf. Theory, 2015

[3] S. Huang et al., Solving Complex Quadratic Systems with Full-Rank Random Matrices, arXiv, vol.abs/1902.05612, 2019